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SECURITY CLASSIFICATION OF THIS PAGE

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REPORT DOCUMENTATION PAGE AD-A225 960

Form Approved OMB No 0704-0188

DTIC ELECTE AUG 3 0 1990 D

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY N/A	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 302	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 90 0848
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina Center for Stochastic Processes; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code) Statistics Department CB #3260, Phillips Hall Chapel Hill, NC 27599-3260	7b. ADDRESS (City, State, and ZIP Code) Bldg. 410 Bolling Air Force Base, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR; 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620 85C 0144
8c. ADDRESS (City, State, and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO 6.1102F; PROJECT NO 2304; TASK NO A5; WORK UNIT ACCESSION NO

11. TITLE (Include Security Classification) On the spectral SLLN and pointwise ergodic theorem in L^{α}

12. PERSONAL AUTHOR(S) Houdre, C.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
preprint	FROM ______ TO ______	1990, July	23

16. SUPPLEMENTARY NOTATION None.

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	N/A
XXXXXX	XXXXXXXXXX	XXX	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Professor ~~Eytan Barouch~~ Jon A Sjogren	22b. TELEPHONE (Include Area Code) (202)767-5026	22c. OFFICE SYMBOL AFOSR/NM

DD Form 1473, JUN 86 Previous editions are obsolete.

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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON THE SPECTRAL SLLN AND POINTWISE ERGODIC THEOREM IN L^{α}

by

Christian Houdré

Technical Report No. 302

July 1990

ON THE SPECTRAL SLLN AND POINTWISE ERGODIC THEOREM IN L^α

Christian Houdré*

Center for Computational Statistics and Probability
George Mason University
Fairfax, VA 22030–4444

and

Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, NC 27599–3260

Abstract. We obtain conditions for the SLLN to hold for some classes of processes with finite α^{th}–moment, $1 < \alpha \leq 2$, which in addition are Fourier transforms of random measures. With this spectral approach, we also give conditions for the pointwise ergodic theorem to hold, for some classes of operators between L^α–spaces, $1 < \alpha < +\infty$. In particular, we find a criterion which applies to invertible linear operators T on L^2 such that $\sup_{n \in \mathbb{Z}} \|T^n\| < +\infty$. Some random fields extensions are also studied.

Research supported by the ONR Contract No. N00014 86 C 0227 and partially by the AFSOR No. F49620 85C 0144.

*Present address (September 1990): Department of Mathematics, University of Maryland, College Park, MD 20742.

1. Introduction

The criterion, obtained by V. F. Gaposhkin [G1], for a (weakly) stationary process to satisfy the strong law of large numbers (SLLN) has had various extensions, in particular to second order non-stationary harmonizable processes, ([G2], [R], [D]). Outside of the L^2-framework, it has also been studied for Fourier transforms of independently scattered symmetric α-stable (SαS) measures in [CHW]). It is shown here that via this spectral approach, neither the L^2-requirement nor any distributional assumption are indispensable in establishing the SLLN. Only the harmonic representation with respect to a bounded (in a sense to be made precise) random measure is crucial. This is illustrated in the present work, where we obtain conditions for the SLLN to hold for some classes of processes with finite α^{th}-moment which, in addition, are Fourier transforms.

It is well known that stationary processes and unitary groups of operators are interchangeable, and so are the corresponding strong law and pointwise ergodic theorem. This type of duality between operators and processes carries over to our framework, although in general, the operators are not shifts. It is, thus, also the purpose of our work to obtain the pointwise ergodic theorem, for some new classes of operators between L^{α}-spaces, $1 < \alpha < +\infty$.

We now give a brief description of the content of this paper. In the next section, we set the stage. We introduce the processes under study and also illustrate the scope of our approach with various examples. Section 3 is the core of the paper, and ergodic properties of processes are developed. These results recover some classical strong laws such as the ones for orthogonal random variables, martingale difference processes, etc.. In section 4, we adapt our framework to operators to give a criterion for the pointwise ergodic theorem to hold for some new classes of operators between L^{α}-spaces. This also recovers some classical results. In the last section, we discuss some random fields generalizations.

Acknowledgement: The reinterpretation of our results in an operator theoretic framework,

i.e. Section 4, stemmed from conversations with Karl Petersen for which the author is heartily thankful.

2. Preliminaries

Let $(\Omega,\mathcal{B},\mathcal{P})$ be a probability space and let $L^\alpha(\Omega,\mathcal{B},\mathcal{P})$ ($L^\alpha(\mathcal{P})$ for short) be the corresponding space of complex valued random variables with finite α^{th}–moment, $1 \le \alpha \le 2$ and let also $L^0(\Omega,\mathcal{B},\mathcal{P})$ be the space of random variables. On $L^\alpha(\mathcal{P})$ the norm, i.e., $(\mathcal{E}|\cdot|^\alpha)^{1/\alpha}$, is denoted by $\|\cdot\|_\alpha$ where $\mathcal{E}$ the expectation. Finally, throughout, K denotes a generic absolute constant whose value might change from an expression to another.

We now recall some terminology and results which are in [H1–H3]. A (strongly) continuous (norm) bounded process $X : \mathbb{R} \longrightarrow L^\alpha(\mathcal{P})$ is *(α,β)-bounded*, $1 \le \beta \le +\infty$, if and only if there exists a finitely additive $Z : \mathcal{B}_0(\mathbb{R}) \longrightarrow L^0(\mathcal{P})$ of bounded (α,β)–variation such that $X_t = \lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} (1-\frac{|\xi|}{\lambda})e^{it\xi}dZ(\xi)$ in $L^\alpha(\mathcal{P})$, uniformly on compact subsets of $\mathbb{R}$, and where $\mathcal{B}_0(\mathbb{R})$ are the Borel sets with finite Lebesgue measure. For $\beta = +\infty$, Z is σ–additive on the Borel sets $\mathcal{B}(\mathbb{R})$, the exponentials become Z–integrable, and $\lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} (1-\frac{|\xi|}{\lambda})e^{it\xi}dZ(\xi) = \int_{\mathbb{R}} e^{it\xi}dZ(\xi)$. For $\alpha = 2$, (α,∞)–bounded processes are also known as *(weakly) harmonizable* and when $\mathcal{E}Z(\cdot)Z(\cdot) : \mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{C}$, extends to a measure on $\mathcal{B}(\mathbb{R}^2)$ they are *(Loève or strongly harmonizable)*. For Z *orthogonally scattered*, i.e., $\mathcal{E}Z(A)\overline{Z(B)} = 0$ whenever $A \cap B = \phi$, $A, B \in \mathcal{B}(\mathbb{R})$, the process X is stationary. For $1 < \alpha < 2$, a typical example of random measure Z is an independently scattered isotropic SαS with finite control measure in which case, Z has finite (p,∞)–variation for any $1 \le p < \alpha$. To illustrate the scope and the applicability of our results we now present some less typical examples (we do consider the case $\alpha > 2$ for future considerations on operators given in Section 4). A discrete time orthogonal process $X = \{X_n\}$ with $\mathcal{E}|X_n|^2 \le K$ is (2,2)–bounded (see [H1]). By taking i.i.d. Bernoulli (or Rademacher) random variables via

Khintchine inequality we can get $(\alpha,2)$–boundednesss, $1 \leq \alpha < +\infty$. For less "independent" examples, let $\alpha \geq 2$ and let $X : \mathbb{Z} \longrightarrow L^{\alpha}(\mathcal{P})$ be a norm bounded martingale difference process. Then, by Burkholder's and Minkowski's inequality we have

$$\begin{aligned}\|\sum_{i=1}^{N} P_i X_{n_i}\|_{\alpha}^{\alpha} &\leq K \|\sum_{i=1}^{N} |P_i|^2 |X_{n_i}^2|\|_{\alpha/2}^{\alpha/2} \\ &\leq K \{\sum_{i=1}^{N} |P_i|^2 \|X_{n_i}^2\|_{\alpha/2}\}^{\alpha/2} \\ &\leq K \sup \|X_{n_i}\|_{\alpha}^{\alpha} \{\sum_{i=1}^{N} |P_i|^2\}^{\alpha/2}.\end{aligned}$$

Again, X is $(\alpha,2)$–bounded, but for $\alpha \geq 2$, hence, $X_n = \int_{-\pi}^{\pi} e^{in\xi} dZ(\xi)$, $n \in \mathbb{Z}$, where Z is "dominated" by Lebesgue measure. In all these examples, X and Z can be recovered from one another by inversion formulae.

Since a (α,β)–bounded process X is strongly continuous, we can assume that it is (t,ω)–measurable with locally integrable sample paths and the averages $\sigma_T X(\omega) = \frac{1}{2T}\int_{-T}^{T} X(t,\omega)dt$, $T > 0$, $\omega \in \Omega$, are well defined. We then say that X satisfies the SLLN whenever $\lim_{T\to\infty} \sigma_T X(\omega) = 0$ ($\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} X_n(\omega) = 0$ in the discrete time case), for almost all ω (we will usually omit the reference to ω, e.g., write $\sigma_T X$ for $\sigma_T X(\omega)$).

We now state two majorizing lemmas which are needed thereafter. The proof of the first one is a zero complexity extension (replacing Cauchy–Schwarz's inequality by Hölder's inequality) of a result of Rousseau [R], which goes back to Gàl and Koksma [GK], and so is omitted. The second one, which for $\alpha = 2$ is just one of the various form of the famous Grothendieck's inequality, can be found in Pisier [P] and [H3].

We need some more notation. Given any integer $p \geq 0$, an integer $n \geq 2$ such that $2^p < n \leq 2^{p+1}$ has a unique binary decomposition $n = 2^p + 1 + \sum_{j=1}^{p} \epsilon_j 2^{p-j}$, where $\epsilon = (\epsilon_1,\ldots,\epsilon_p) \in \{0,1\}^p$. Hence, to any such n, i.e., to any sequence $\epsilon \in \{0,1\}^p$ we can also associate the (finite) sequence

$$a_k(\epsilon,p) = \begin{cases} 2^p + 1 + \sum_{j=1}^{k} \epsilon_j 2^{p-j}, & k=1,2,\dots,p \\ 2^p & k=0 \end{cases} .$$

With these notations and if a_k is short for $a_k(\epsilon,p)$, we have.

Lemma 2.1. Let $\{z_j\}$ be a sequence of complex numbers and let $\{t_j\}$ be a sequence of positive numbers. Then, for any $p \geq 1$ and $1 < \alpha \leq 2$, we have

$$\max_{2^p<n\leq 2^{p+1}} \Big| \sum_{j=2^p+1}^{n} z_j \Big|^{\alpha} \leq \Big(\sum_{k=1}^{p} t_k^{(1-\alpha)^{-1}} \Big)^{\alpha-1} \Big(\sum_{k=1}^{p} t_k \sum_{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} \Big| \sum_{j=a_{k-1}+1}^{a_k} z_j \Big|^{\alpha} \Big).$$

Lemma 2.2. Let the process X be norm bounded, strongly continuous and (α,∞)–bounded, $1 \leq \alpha \leq 2$, with associated random measure Z. Then, there exists a finite positive measure ν such that

$$\Big\| \int_{\mathbf{R}} f dZ \Big\|_{\alpha} \leq \Big(\int_{\mathbf{R}} |f|^2 d\nu \Big)^{1/2}, \tag{2.1}$$

for all $f \in L^2(\nu)$ (the functions which are square–integrable with respect to $d\nu$).

It is clear, that whenever X is strongly continuous and (α,β)–bounded, the Fourier representation as well as the boundedness property give $\lim_{T\to\infty} \sigma_T X = Z(0)$, in $L^{\alpha}(\mathcal{P})$. But, it is also well know that, even in the stationary case, $Z(0) = 0$ a.s., is not a sufficient condition for the SLLN to hold. Similarly, it is not because the dominating ν is (2.1) is, say, the spectral measure of a stationary process satisfying the SLLN, that the dominated X satisfies the SLLN. After all, (2.1) is just a norm estimate. However, this norm estimate is a strong ingredient in obtaining a criterion for the almost sure convergence of the ergodic averages.

To finish our preliminaries, we say that a dominating measure ν in (2.1) is

$\alpha/2$-atomistic if $\sum_{k=0}^{\infty} \nu\{2^{-k-1}<|\xi|\leq 2^{-k}\}^{\alpha/2} < +\infty$. It is clear that for $\alpha = 2$, ν is always $\alpha/2$-atomistic and such is also the case for $1 \leq \alpha < 2$ when $d\nu(t) = \nu(t)dt$, $\nu \in L^{\gamma}(\mathbb{R})$, $1 < \gamma \leq +\infty$, or when ν is discrete with jumps j_k such that $\sum_{k=0}^{\infty} j_k^{\alpha/2} < +\infty$.

3. The Spectral SLLN

With the result of the previous section, our approach in proving the SLLN follows classical paths. The first of which is another lemma showing that we can reduce the problem to the dyadic subsequences. We prove our results only for the more interesting situation of continuous time processes with also $\beta = +\infty$, discrete time result are obtainable in a identical fashion. The case $\beta < +\infty$ will be worth a separate statement (Theorem 3.6). *Finally, throughout this section, and unless otherwise stated, (α,∞)-bounded is short for strongly continuous, norm bounded, (α,∞)-bounded, with also $1 < \alpha \leq 2$.*

Lemma 3.1. Let X be (α,∞)-bounded with $\alpha/2$-atomistic dominating measure, then $\lim_{p\to+\infty} \max_{2^p < n \leq 2^{p+1}} |\sigma_n X - \sigma_{2^p} X| = 0$, (a.s. $\mathcal{P}$).

Proof. As in the harmonizable case (strong or weak, see Rousseau [R] or Dehay [D]), and since $\sigma_n X - \sigma_{2^p} X = \sum_{j=2^p+1}^{n} (\sigma_j X - \sigma_{j-1} X)$, applying Lemma 2.1 with its notation, we get

$$\mathcal{E} \max_{2^p < n \leq 2^{p+1}} |\sigma_n X - \sigma_{2^p} X|^{\alpha}$$

$$\leq \Big(\sum_{k=1}^{p} t_k^{(1-\alpha)^{-1}}\Big)^{\alpha-1} \Big(\sum_{k=1}^{p} t_k 2^k \max_{(\epsilon_1,\ldots,\epsilon_k)\in\{0,1\}^k} \mathcal{E}|\sigma_{a_k} X - \sigma_{a_{k-1}} X|^{\alpha}\Big). \tag{3.1}$$

To prove the result, it is enough to show that $\mathcal{E} \max_{2^p < n \leq 2^{p+1}} |\sigma_n X - \sigma_{2^p} X|^{\alpha}$ is the general term of a convergent series. Since X is (α,∞)–bounded, for any $T > 0$ we have $\sigma_T X = \int_{\mathbb{R}} (\frac{\sin T\xi}{T\xi}) dZ(\xi)$ and by (3.1) and (2.1), it is in turn enough to show that

$$\left(\sum_{k=1}^{p} t_k^{(1-\alpha)^{-1}}\right)^{\alpha-1}\left(\sum_{k=1}^{p} t_k 2^k \max_{\epsilon\in\{0,1\}^k} \left\{\int_{\mathbb{R}} \left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 d\nu(\xi)\right\}^{\alpha/2}\right), \quad (3.2)$$

is itself the general term of convergent series. To do so, and as for stationary or harmonizable processes, we divide $\mathbb{R}$ into four pieces, $\{|\xi| \le 2^{-p-1}\}$, $\{2^{-p-1} < |\xi| \le 2^{-p+k}\}$, $\{2^{-p+k} < |\xi| \le 1\}$, $\{|\xi| > 1\}$. We then use the triangle inequality and proceed to estimate each one of the resulting four sums. The estimates over the four different regions are similar and so we just give the details for, say, the second region. Since $\left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 \le K|a_k - a_{k-1}|^2/|a_{k-1}|^2 \le K2^{(p-k)2}/\,2^{2p}$, we have

$$\left\{\int_{\{2^{-p-1}<|\xi| \le 2^{-p+k}\}} \left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 d\nu(\xi)\right\}^{\alpha/2}$$

$$\le K2^{-\alpha k}\left\{\int_{\{2^{-p-1}<|\xi| \le 2^{-p+k}\}} d\nu(\xi)\right\}^{\alpha/2}.$$

With the previous inequality and taking $t_k = t^k$, $1 < t < 2^{\alpha-1}$, we get

$$\sum_{p=1}^{\infty}\left(\sum_{k=1}^{p} t_k 2^k \max_{\epsilon\in\{0,1\}^k}\left\{\int_{\{2^{-p-1}<|\xi| \le 2^{-p+k}\}}\left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 d\nu(\xi)\right\}^{\alpha/2}\right)$$

$$\le K\sum_{p=1}^{\infty}\left(\sum_{k=1}^{p} t^k 2^k 2^{-\alpha k}\left\{\int_{\{2^{-p-1}<|\xi| \le 2^{-p+k}\}} d\nu(\xi)\right\}^{\alpha/2}\right)$$

$$= K\sum_{p=1}^{\infty}\left(\sum_{k=1}^{p} t^k 2^{(1-\alpha)k}\left\{\sum_{j=p-k}^{p}\int_{\{2^{-j-1}<|\xi| \le 2^{-j}\}} d\nu(\xi)\right\}^{\alpha/2}\right).$$

$$\le K\sum_{p=1}^{\infty}\left(\sum_{k=1}^{p} t^k 2^{(1-\alpha)k}\sum_{j=p-k}^{p} \nu\{2^{-j-1}<|\xi|\le 2^{-j}\}^{\alpha/2}\right), \text{ since } \alpha \le 2.$$

Now, by rearranging its terms the above series becomes

$$= K \sum_{j=0}^{\infty} \nu\{2^{-j-1} < |\xi| \le 2^{-j}\}^{\alpha/2} \sum_{k=1}^{\infty} \sum_{p=\max(j,k)}^{j+k} t^k 2^{(1-\alpha)k}$$

$$\le K \sum_{j=0}^{\infty} \nu\{2^{-j-1} < |\xi| \le 2^{-j}\}^{\alpha/2} \sum_{k=1}^{\infty} (k+1) t^k 2^{(1-\alpha)k} < +\infty, \text{ since } \nu \text{ is } \alpha/2\text{-atomistic.}$$

Finally, $\{\sum_{k=1}^{p} t_k^{(1-\alpha)^{-1}}\}^{\alpha-1} \le \{1-t^{(1-\alpha)^{-1}}\}^{-1}$, and the sum (3.2) with $\mathbb{R}$ replaced by $\{2^{-p-1} < |\xi| \le 2^{-p+k}\}$ is thus finite. For the first region, we use $\left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 \le K|\xi|^2|a_k - a_{k-1}|^2 \le K|\xi|^2 2^{2(p-k)}$ and proceed similarly. The third series can be estimated using $\left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 \le K|a_k - a_{k-1}|/|a_{k-1}||\xi a_{k-1}|^{2-1} \le K2^{(p-k)}2^{-2p}/|\xi|^1$. To estimate the last sum, we use $\left|\frac{\sin a_k\xi}{a_k\xi} - \frac{\sin a_{k-1}\xi}{a_{k-1}\xi}\right|^2 \le K/|a_{k-1}\xi|^2 \le K2^{-2p}/|\xi|^2$. Thus (3.2) is finite, and the result follows. ∎

We now state the main result of this section, a criterion for the SLLN to hold for (α,∞)-bounded processes with $\alpha/2$-atomistic dominating measure.

Theorem 3.2. Let X be (α,∞)-bounded with random measure Z and $\alpha/2$-atomistic dominating measure. The following conditions are equivalent.

(i) For a.a. ω, $\lim_{T\to+\infty} \sigma_T X(\omega)$ exists.

(ii) For a.a. ω, $\lim_{p\to+\infty} Z\{|\xi| < 2^{-p}\}(\omega)$ exists.

Under either condition, and for a.a. ω, $\lim_{T\to\infty} \sigma_T X = \lim_{p\to+\infty} Z\{|\xi| < 2^{-p}\} = Z(0)$.

Proof. Again, as in the L^2-case (stationary or harmonizable)

$$\sigma_T X = (\sigma_T X - \sigma_n X) + (\sigma_n X - \sigma_{2^p} X) + (\sigma_{2^p} X - Z\{|\xi| < 2^{-p}\}) + Z\{|\xi| < 2^{-p}\}.$$

Since $\|X_t\|_\alpha \leq K$, arguments similar to the ones of Proposition 1 in [R] gives $\lim_{n\to\infty} \sup_{n<T\leq n+1} |\sigma_T X - \sigma_n X| = 0$ a.s., and the first parenthesis converges to 0 a.s.. The middle parenthesis is taken care of by the previous lemma. For the third parenthesis, to conclude that $\lim_{p\to+\infty} \sigma_{2^p} X(\omega) - Z\{|\xi| < 2^{-p}\}(\omega) = 0$, a.s., we again wish to show that $\mathcal{E}|\sigma_{2^p} X - Z\{|\xi| < 2^{-p}\}|^\alpha$ is the general term of a convergent series. But, using the triangle inequality as well as Lemma 2.2 we have,

$$\sum_{p=1}^{\infty} \mathcal{E}|\sigma_{2^p} X - Z\{|\xi| < 2^{-p}\}|^\alpha$$

$$\leq \sum_{p=1}^{\infty} \left\{ \left[\int_{\{|\xi|<2^{-p}\}} \left| \frac{\sin 2^p\xi}{2^p\xi} - 1 \right|^2 d\nu(\xi) \right]^{\alpha/2} + \left[\int_{\{|\xi|\geq 2^{-p}\}} \left| \frac{\sin 2^p\xi}{2^p\xi} \right|^2 d\nu(\xi) \right]^{\alpha/2} \right\},$$

To prove the result, it is thus again enough to show that both series converges. We provide the details only for the first integral. Since $\left| \frac{\sin 2^p\xi}{2^p\xi} - 1 \right|^2 \leq K 2^{2p} |\xi|^2$, the first sum is dominated by

$$K \sum_{p=1}^{\infty} 2^{\alpha p} \{ \sum_{k=p}^{\infty} \int_{\{2^{-k-1} \leq |\xi| < 2^{-k}\}} |\xi|^2 d\nu(\xi) \}^{\alpha/2}$$

$$\leq K \sum_{p=1}^{\infty} 2^{\alpha p} (\sum_{k=p}^{\infty} 2^{-2k} \nu\{2^{-k-1} \leq |\xi| < 2^{-k}\})^{\alpha/2}$$

$$\leq K \sum_{p=1}^{\infty} 2^{\alpha p} \sum_{k=p}^{\infty} 2^{-\alpha k} \nu\{2^{-k-1} \leq |\xi| < 2^{-k}\}^{\alpha/2}$$

$$= K \sum_{k=1}^{\infty} 2^{-\alpha k} \nu\{2^{-k-1} \leq |\xi| < 2^{-k}\}^{\alpha/2} \sum_{p=1}^{k} 2^{\alpha p}$$

$$\leq K \sum_{k=1}^{\infty} \nu\{2^{-k-1} \leq |\xi| < 2^{-k}\}^{\alpha/2} + K \sum_{k=1}^{\infty} 2^{-\alpha k} \nu\{2^{-k-1} \leq |\xi| < 2^{-k}\}^{\alpha/2} < +\infty.$$

Using $\left|\frac{\sin 2^p\xi}{2^p\xi}\right|^2 \leq K/|2^p\xi|^2$, the second sum can be estimated in a similar way. Thus, $\lim_{p\to\infty} (\sigma_{2^p} X - Z\{|\xi| < 2^{-p}\}) = 0$ (a.s. $\mathcal{P}$). Finally, (α,∞)–boundedness via Lemma 2.2 give $\lim_{p\to\infty} Z\{|\xi| < 2^{-p}\} = Z(0)$ in $L^\alpha(\mathcal{P})$, and the result follows. ∎

Before restating Theorem 3.2, in an essentially equivalent way, we need to introduce some more notation. Let $c_k = \|\int_{\{2^{-k-1} \leq |\xi| < 2^{-k}\}} dZ(\xi)\|_\alpha$, and let $Z_k = c_k^{-1}\{\int_{\{2^{-k-1} \leq |\xi| < 2^{-k}\}} dZ(\xi)\}$ for $c_k > 0$ and $Z_k = 0$ when $c_k = 0$, $k \geq 1$.

Theorem 3.3. Let X be (α,∞)–bounded with random measure Z and $\alpha/2$–atomistic dominating measure. Then, X satisfies the SLLN if and only if one of the following equivalent conditions holds.

(i) For a.a. ω, $\lim_{p\to+\infty} Z\{|\xi| < 2^{-p}\}(\omega) = 0$.

(ii) For a.a. ω, $\lim_{p\to+\infty} \sum_{k=1}^{p} c_k Z_k$ exists, and $Z(0) = 0$.

Proof. In $L^\alpha(\mathcal{P})$ and by (α,∞)–boundedness, we have $\int_{\{0 < |\xi| < 2^{-p}\}} dZ(\xi) = \sum_{k=p}^{\infty} \int_{\{2^{-k-1} \leq |\xi| < 2^{-k}\}} dZ(\xi) = \sum_{k=p}^{\infty} c_k Z_k$. Since by (α,∞)–boundedness, $Y = \sum_{k=1}^{\infty} c_k Z_k \in L^\alpha(\mathcal{P})$, we also have $\int_{\{|\xi| < 2^{-p}\}} dZ(\xi) = Z(0) + Y - \sum_{k=1}^{p-1} c_k Z_k$ (in $L^\alpha(\mathcal{P})$), and the result follows from the previous theorem. ∎

Remark 3.4. In the above results, 2 can be replaced by any integer ≥ 2. The $\alpha/2$–atomistic requirement is also not minimal since for "harmonizable" non stationary stable processes, the condition: $\sum_{k=0}^{\infty} \nu\{2^{-k-1} < |\xi| \leq 2^{-k} \times]-\pi,\pi]\}^{p/\alpha} < +\infty$, $1 < p < \alpha$, where ν is the finite two dimensional "spectral" measure corresponding to the independently scattered SαS random measure, will also do the job. We note too that for a general independently scattered Z, the above Z_k are independent random variables. Hence, since $\lim_{N\to+\infty} \sum_{k=1}^{N} c_k Z_k = Z(0)$ (in $L^\alpha(\mathcal{P})$) and since by independence convergence in probability and almost surely of the series are the same, we have for a.a. ω, $\lim_{p\to+\infty} Z\{|\xi| < 2^{-p}\}(\omega) = Z(0)$. Again, in the non-stationary stable harmonizable case, $\sum_{k=0}^{\infty} \nu\{2^{-k-1} < |\xi| \le 2^{-k} \times]-\pi,\pi]\}^{p/\alpha} < +\infty$, $1 < p < \alpha$, and $Z(0) = 0$ gives the SLLN. We do not know, however, if the mere condition $Z(0) = 0$, is sufficient to give the SLLN. Finally, by analyzing the above proofs, it is also easily seen that under the less stringent requirement: $\sum_{p=1}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-2k} \nu\{2^{-p-k-1} < |\xi| \le 2^{-p-k}\}\right)^{\alpha/2} < +\infty$, Theorem 3.2 and 3.3 continue to hold. Again, we note that in the stable case this last condition can, as above, be replaced by $\sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-2k} \nu\{2^{-n-k-1} < |\xi| \le 2^{-n-k}\} \times]-\pi,\pi]\right)^{p/\alpha} < +\infty$, $1 < p < \alpha$.

In our framework, the sufficient conditions given by Gaposhkin in the stationary case or by Dehay in the weakly harmonizable case for the strong law to hold, become:

Theorem 3.5. Let X be (α,∞)–bounded with $\alpha/2$–atomistic dominating measure with also $\|Z(0)\|_\alpha = 0$. If there exists a finite positive measure ν on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that

(i) $\|Z(A)\|_\alpha^\alpha \le \nu(A\times A)$, $A \in \mathcal{B}_0(\mathbb{R})$

(ii) $\iint_{\{0<|\xi|,|\eta|<\delta\}} (\log_2\log_2 \frac{1}{|\xi|})^{\alpha/2} (\log_2\log_2 \frac{1}{|\eta|})^{\alpha/2} d\nu(\xi,\eta) < +\infty$, for some $\delta > 0$.

Then, X satisfies the SLLN.

Proof. To prove the assertion, it is enough to show that the sequence $Z\{|\xi| < 2^{-p}\}$ converges with probability 1, and to do so, we again show that $\mathcal{E}|Z\{0 < |\xi| < 2^{-p}\}|^\alpha$ is the general term of a convergent series. Our proof is only sketched since similar to the one devised by Dehay [D], for $\alpha = 2$. First, for $2^q < p \le 2^{q+1}$, we have $Z\{|\xi| < 2^{-p}\} = Z(0)$

$+ Z\{0 < |\xi| < 2^{-2^q}\} - Z\{2^{-p} \le |\xi| < 2^{-2^q}\}$. But, by (i) and (ii) above with q_0 any integer such that $2^{-2^{q_0}} < \delta$, we have if $A_q = \{0 < |\xi| < 2^{-2^q}\}$ and, if log denotes the logarithm of base 2:

$$\sum_{q=q_0}^{\infty} \|Z(A_q)\|_\alpha^\alpha \le \sum_{q=q_0}^{\infty} \nu(A_q \times A_q)$$

$$\le \sum_{q=q_0}^{\infty} q^{-\alpha} \iint_{A_q \times A_q} (\operatorname{loglog} \tfrac{1}{|\xi|})^{\alpha/2} (\operatorname{loglog} \tfrac{1}{|\eta|})^{\alpha/2} d\nu(\xi,\eta)$$

$$\le \iint_{\{0<|\xi|,|\eta|<\delta\}} (\operatorname{loglog} \tfrac{1}{|\xi|})^{\alpha/2} (\operatorname{loglog} \tfrac{1}{|\eta|})^{\alpha/2} d\nu(\xi,\eta) \sum_{q=q_0}^{\infty} q^{-\alpha} < +\infty.$$

Hence, $\lim_{r\to+\infty} Z\{0<|\xi|<2^{-2^q}\} = 0$ a.s. For $Z\{2^{-p} \le |\xi| < 2^{-2^q}\}$, let $B_k = \{2^{-a_k} \le |\xi| < 2^{-a_{k-1}}\}$ where a_k is defined as before, and let $C_q = \{2^{-2^{q+1}} \le |\xi| < 2^{-2^q}\}$. Applying Lemma 2.1 with $t_k = 1$ as well as (i) and (ii) we get

$$\sum_{q=q_0}^{+\infty} \mathcal{E}\Big(\max_{2^q<p\le 2^{q+1}} |Z\{2^{-p} \le |\xi| < 2^{-2^q}\}|^\alpha\Big)$$

$$\le \sum_{q=q_0}^{+\infty} q^{\alpha-1}\Big(\sum_{k=1}^{q} \sum_{(\epsilon_1,\ldots,\epsilon_k)\in\{0,1\}^k} \|Z(B_k)\|_\alpha^\alpha\Big)$$

$$\le \sum_{q=q_0}^{+\infty} q^{\alpha-1}\Big(\sum_{k=1}^{q} \nu(C_k \times C_k)\Big) \le \sum_{q=q_0}^{+\infty} q^\alpha \nu(C_q \times C_q)$$

$$\le \iint_{\{0<|\xi|,|\eta|<\delta\}} (\operatorname{loglog} \tfrac{1}{|\xi|})^{\alpha/2} (\operatorname{loglog} \tfrac{1}{|\eta|})^{\alpha/2} d\nu(\xi,\eta) < +\infty.$$

Hence, with probability one, $\lim_{q\to+\infty} \max_{2^q<p\le 2^{q+1}} Z\{2^{-p} \le |\xi| < 2^{-2^q}\} = 0.$ ■

Remark 3.6. The construction given by Feder [F] can be easily adapted to give a discrete time $(\alpha,2)$-bounded (hence (α,∞)-bounded) process X such that $\sigma_{2^p}X$ diverges on a arbitrary set of positive measure. In fact, such X can be chosen with absolutely continuous dominating measure $d\nu(t) = \nu(t)dt$, $\nu \in L^1(]-\pi,\pi])$ by also adapting the arguments in [G1].

We now present the SLLN for (α,β)-bounded processes when $1 < \beta < +\infty$, for which the results are simpler.

Theorem 3.7. Let X be (α,β)-bounded, $\beta < +\infty$, then X satisfies the SLLN.

Proof. From the very definition of (α,β)-boundedness (see [H3]), when $\beta < +\infty$ we have $\|\int_{\mathbb{R}} f dZ\|_\alpha \leq (\int_{\mathbb{R}} |f(t)|^\beta dt)^{1/\beta}$, for all $f \in L^\beta(\mathbb{R})$, hence, $\|Z(0)\|_\alpha = 0$. Furthermore X is the uniform limit on compact sets of Cesàro averages, hence for any $T > 0$ we have $\sigma_T X = \lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} (1-\frac{|\xi|}{\lambda}) \frac{\sin T\xi}{T\xi} dZ(\xi) = \int_{\mathbb{R}} (\frac{\sin T\xi}{T\xi}) dZ(\xi)$ (Z has finite (α,β)-variation, $\beta > 1$, hence $L^\beta(\mathbb{R})$-functions are Z-integrable and $\frac{\sin T\xi}{T\xi} \in L^\beta(\mathbb{R})$). Now, repeating the steps in Theorem 3.2 and 3.3, using also Hölder's inequality give the result. ∎

Remark 3.8. When $\alpha = \beta = 2$, e.g., for martingale difference or orthogonal processes, our results are not optimal, in the sense that the rate of convergence can be improved. By techniques similar to the ones in Theorem 3.7, or via Theorem 3.3 (ii) with an appropriate extension of the classical Rademacher-Menchov theorem it can be shown that with probability one, $\lim_{N\to\infty} \frac{1}{\sqrt{2N+1}\,\mathrm{Log}^{3+\epsilon}(2N+1)} \sum_{n=-N}^{N} X_n(\omega) = 0$, $\epsilon > 0$. For $\beta = +\infty$, it can also be shown with the above techniques and as in [G1], [G2], [R]. and [D] for $\alpha = 2$, that $\lim_{T\to\infty} (\mathrm{LogLog}2T)^{-\alpha/2} \sigma_T X = 0$, with probability one. The LogLog speed being the best possible. The various necessary or sufficient conditions given in the works mentioned above admit also α-counterparts. Finally, for stationary processes, Theorem 3.7 corresponds to the SLLN for processes with spectral densities $f \in L^{1+\epsilon}(\mathbb{R})$, $\epsilon > 0$.

4. The Pointwise Ergodic Thoorem

In this section, we now turn our attention to operators and introduce first some definitions which parallel the corresponding notions for processes. Throughout this section we replace our probability space $(\Omega,\mathcal{B},\mathcal{P})$ by a measure space (also denoted $(\Omega,\mathcal{B},\mathcal{P})$), and we extend the range α beyond 2 and assume that $1 < \alpha < +\infty$.

Let $B(L^{\alpha})$ be the algebra of bounded linear operators on $L^{\alpha}(\mathcal{P})$ equipped with the strong operator topology and let $\|\cdot\|$ denote the usual norm on $B(L^{\alpha})$. Throughout, let also $T : \mathbb{R} \longrightarrow B(L^{\alpha})$ be an operator function, i.e., let T be bounded ($\sup_t \|T^t g\|_{\alpha} \leq K$, $g \in L^{\alpha}(\mathcal{P})$) and measurable ($t \longrightarrow T^t g$ is strongly measurable). Since T is bounded, $\int_{\mathbb{R}} f(t)T^t dt$ is a well defined Lebesgue–Bochner integral for any $f \in L^1(\mathbb{R})$. Recalling that for $\beta < +\infty$ (resp. $\beta = +\infty$), $\|\cdot\|_{\beta}$ is the norm on $L^{\beta}(\mathbb{R})$ (resp. on $C_0(\mathbb{R})$) we set.

Definition 4.1. An operator function T is *(α,β)–bounded* if there exists $K > 0$ such that

$$\left\|\int_{\mathbb{R}} f(t)T^t dt\right\| \leq K \|\hat{f}\|_{\beta},$$

for all $f \in L^{\beta}(\mathbb{R})^{\vee} = \{ f \in L^1(\mathbb{R}) : \hat{f} \in L^{\beta}(\mathbb{R}) \}$ ($f \in L^1(\mathbb{R})$ when $\beta = +\infty$).

A function $E : \mathcal{B}_0(\mathbb{R}) \longrightarrow B(L^{\alpha})$ is called an *operator measure* whenever it is finitely additive and a *spectral measure* if in addition it is multiplicative, i.e., $E(A \cap B) = E(A)E(B)$, $A, B \in \mathcal{B}_0(\mathbb{R})$. The relation between operator and spectral measures is easy to draw: an operator measure is a spectral measure if and only if it is projection valued (this can be proved as in Helson [He]). Clearly, these projections are also commuting projections.

Definition 4.2. An operator measure E has finite *(α,β)–variation* if $|||E||| = \sup\{ |||E|||(A): A \in \mathcal{B}_0(\mathbb{R})\} < +\infty$, where $|||E|||(A) = \sup\{ \|\sum_{i=1}^{N} a_i E(A_i)\| : \{A_i\}_1^N \subset \mathcal{B}_0(\mathbb{R})$ partition of A, $a_i \in$

$\mathbb{C}$, $\|\sum_{i=1}^{N} a_i \chi_{A_i}\|_\beta \leq 1$ }.

For any $g \in L^\alpha(\mathcal{P})$, E_g given by $E_g(A) = E(A)g$, $A \in \mathcal{B}_0(\mathbb{R})$, defines a random measure. Furthermore, by uniform boundedness, $|||E|||(A) < +\infty$ if and only if for every $g \in L^\alpha(\mathcal{P})$, $|||E_g|||(A) < +\infty$ where $|||E_g|||(A) = \sup\{ \|\sum_{i=1}^{N} a_i E(A_i)g\|_\alpha : \{A_i\}_1^N \subset \mathcal{B}_0(\mathbb{R})$ partition of A, $a_i \in \mathbb{C}$, $\|\sum_{i=1}^{N} a_i \chi_{A_i}\|_\beta \leq 1$ }. The integral $\int_{\mathbb{R}} f(\xi)dE(\xi)$ of the scalar function f with respect to the operator measure E of bounded (α,β)-variation can now be defined as the element of $B(L^\alpha)$ for which $(\int_{\mathbb{R}} f(\xi)dE(\xi))g = \int_{\mathbb{R}} f(\xi)dE_g(\xi)$, $g \in L^\alpha(\mathcal{P})$. A more direct definition, without any appeal to the random measures E_g, can also be given using the norm $\|\cdot\|$. For E of bounded (α,β)-variation ($\beta < +\infty$), any f in $L^\beta(\mathbb{R})$ is integrable with respect to E, while for $\beta = +\infty$, the Borel bounded functions are also E-integrable. Moreover, $\|\int_{\mathbb{R}} f(\xi)dE(\xi)g\|_\alpha \leq |||E|||\, \|f\|_\beta \|g\|_\alpha$, for $f \in L^\beta(\mathbb{R})$ and $g \in L^\alpha(\mathcal{P})$.

With the above definitions we can state our first result.

Theorem 4.3. An operator function T is continuous and (α,β)-bounded if and only if there exists a (unique regular) operator measure E with finite (α,β)-variation such that $T^t = \lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} e^{it\xi}dE(\xi)$ (in $B(L^\alpha)$ with the strong operator topology), uniformly on compact subsets of $\mathbb{R}$.

Proof. Let T be continuous and (α,β)-bounded, then for any $g \in L^\alpha(\mathcal{P})$, $\{T^t g\}_{t\in\mathbb{R}}$ is an (α,β)-bounded strongly continuous process. Thus, [H3, Theorem 3.2] (actually Theorem 3.2 there is stated in terms of the triangular kernel, but the proof carries over to the step kernel case) there exists a (unique regular) random measure Z_g with finite (α,β)-variation such that $T^t g = \lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} e^{it\xi}dZ_g(\xi)$, in $L^\alpha(\mathcal{P})$, uniformly on compact subsets of $\mathbb{R}$. Moreover, $|||Z_g||| \leq K$ and $\|Z_g(A)\|_\alpha \leq K\|g\|_\alpha$, for every $A \in \mathcal{B}_0(\mathbb{R})$. Hence, $E : \mathcal{B}_0(\mathbb{R}) \longrightarrow$

$\mathcal{B}(L^\alpha)$ defined via $E(A)g = Z_g(A)$, $g \in L^\alpha(\mathcal{P})$ satisfies all the stated requirements. For the converse, let $T^t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi} dE(\xi)$ in the strong operator topology, uniformly on compact subsets of $\mathbb{R}$. Then, again by Theorem 3.2 of [H3] and for any $g \in L^\alpha(\mathcal{P})$, $\{T^t g\}_{t \in \mathbb{R}}$ is strongly continuous with moreover $\|\int_{\mathbb{R}} f(t) T^t g dt\|_\alpha \le |||E_g|||\ \|\hat{f}\|_\beta$. Since $|||E_g||| \le |||E|||\ \|g\|_\alpha$, the result follows. ∎

Remark 4.4. Since $L^\alpha(\mathcal{P})$ is weakly complete, we were able to replace the relative weak compactness of the sets $\{\|\int_{\mathbb{R}} f(t) T^t g dt\|_\alpha : \|\hat{f}\|_\beta \le 1, f \in L^\beta(\mathbb{R})^\vee$ ($f \in L^1(\mathbb{R})$ when $\beta = +\infty$)$\}$, $g \in L^\alpha(\mathcal{P})$, by their boundedness. For $\beta = +\infty$, E can be defined on $\mathcal{B}(\mathbb{R})$ and is also σ-additive (in the strong operator topology). Hence, by dominated convergence and since the exponentials are E-integrable we have $\lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi} dE(\xi) = \int_{\mathbb{R}} e^{it\xi} dE(\xi)$, $t \in \mathbb{R}$, and this recovers a result of Kluvánek [K]. For $\alpha = 2$ and $\beta = +\infty$, T will be called *harmonizable* even *strongly harmonizable* whenever $\mathcal{E} E_g(\cdot) \bar{E}_g(\cdot)$ can be extended to a complex measure on $\mathbb{R}^2$.

Corollary 4.5. Let T be continuous and (α,β)-bounded with associated operator measure E. Then, T is additive, i.e., $T^{t+s} = T^t T^s$ for all $t, s \in \mathbb{R}$, if and only if E is multiplicative.

Proof. Let E be multiplicative, then for any simple functions f_1 and f_2 with bounded support, $\int_{\mathbb{R}} f_1 f_2 dE = \int_{\mathbb{R}} f_1 dE \int_{\mathbb{R}} f_2 dE$. By (α,β)-boundedness, this equality can be extended to Borel bounded functions, since they can be uniformly approximated by simple functions. It thus follows that $T^{t+s} = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{i(t+s)\xi} dE(\xi) = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi} dE(\xi) \int_{-\lambda}^{\lambda} e^{is\xi} dE(\xi)$ $= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{it\xi} dE(\xi) \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} e^{is\xi} dE(\xi)$, since each individual limit exists. Hence, $T^{t+s} = T^t T^s$. For the converse, since $T^{t+s} = T^t T^s$, and since the trigonometric polynomials of period 2P are uniformly dense in $C([-P,P])$, we have for any f_1, f_2 continuous with compact support $[-P,P]$ and for λ large enough $\int_{\mathbb{R}} f_1 f_2 dE = \int_{\mathbb{R}} f_1 dE \int_{\mathbb{R}} f_2 dE$. Hence, for any Borel bounded f_1 and f_2 (by the domination property, this is immediate for $\beta < +\infty$). Now, we can approximate a.s. a Borel bounded function by a bounded sequence of continuous functions with compact support. Hence, For $\beta = +\infty$, since E can be chosen to be σ-additive on $\mathcal{B}(\mathbb{R})$, the dominated convergence theorem for vector measures will allow us to conclude. ∎

For $\alpha = 2$, when E is orthogonal projection valued, i.e., when for every $A \in \mathcal{B}_0(\mathbb{R})$ $E(A)$ is Hermitian, T is not only additive but also *unitary*, namely, $T^t T^{t*} = T^{t*} T^t = I$ (I is the identity operator). While, the martingale difference case corresponds to operator measures whose values are differences of increasing orthogonal projections.

In general, and in contrast to unitary operators, (α,β)-bounded groups T (T is additive with $T^0 = I$) are not shifts. This can be seen as follows. Let $X = \{X_n\}_{n \in \mathbb{Z}}$ be a discrete time (α,∞)-bounded process, $1 \leq \alpha < +\infty$, $X_n = \int_{-\pi}^{\pi} e^{in\xi} dZ(\xi)$, and let Z be of bounded variation. Then, for any trigonometric polynomial P, $\|\int_{-\pi}^{\pi} P(\theta) dZ(\theta)\|_\alpha \leq \int_{-\pi}^{\pi} |P(\theta)| d|Z|(\theta)$ where $|Z|$ is the total variation measure. It is then not difficult to see (as proved below) that X has a well defined shift if and only if the following condition holds: if for some P, $\|\int_{-\pi}^{\pi} P(\theta) dZ(\theta)\|_\alpha = 0$, then $\int_{-\pi}^{\pi} |P(\theta)| d|Z|(\theta) = 0$. But for $dZ(\theta) = Z\{\chi_{]-\pi,0]}(\theta) - \chi_{]0,\pi]}(\theta)\} d\theta$, where $Z \in L^\alpha(\mathbb{P})$, $Z \neq 0$, this cannot happen unless $P = 0$. To prove the above claim, i.e., to verify that the shift is a well defined operator we need to show that the above stated condition and Getoor's [G] (C_1) condition are the same. Let (C_1) be satisfied, then $\|\int_{-\pi}^{\pi} P(\theta) dZ(\theta)\|_\alpha = 0$ gives $\|\int_{-\pi}^{\pi} e^{in\theta} P(\theta) dZ(\theta)\|_\alpha = 0$ for all $n \in \mathbb{Z}$. Hence, by the uniqueness of the Fourier transform, $PdZ = 0 = |P| d|Z|$. Hence

$\int_{-\pi}^{\pi} |P|d|Z| = 0$. Now if $\|\int_{-\pi}^{\pi} P(\theta)dZ(\theta)\|_\alpha = 0$, implies $\int_{-\pi}^{\pi} |P(\theta)|d|Z|(\theta) = 0$. We have $\|\int_{-\pi}^{\pi} e^{in\theta}P(\theta)dZ(\theta)\|_\alpha \le \int_{-\pi}^{\pi} |e^{in\theta}P(\theta)|d|Z|(\theta) = 0$, and Getoor's (C_1) condition is verified.

After these preliminaries, we can now state the main result of this section. Again, we say that T satisfies the pointwise ergodic theorem whenever for any $g \in L^\alpha(P)$ the averages $\frac{1}{2T}\int_{-T}^{T} T^t g(\omega)dt$, $T > 0$, $\omega \in \Omega$, converge a.s. (P), with of course in discrete time, the integral replaced by $\frac{1}{2N+1}\sum_{n=-N}^{N} T^n g(\omega)$. We only state the result for $2 < \alpha < +\infty$, since the case $\alpha \le 2$ is already in section 3 and since the corresponding statement is slightly different. For $2 < \alpha < +\infty$, the dominating inequality (2.1) becomes $\|\int_{\mathbb{R}} f dE_g\|_\alpha \le \|g\|_\alpha(\int_{\mathbb{R}} |f|^{\alpha+\epsilon}d\nu)^{1/\alpha+\epsilon}$, $\epsilon > 0$ (see [P], the result there is actually not given for $\mathbb{R}$ but for a compact space and also not for operator measures, however, with arguments as in [H3] the above stated inequality can be obtained).

Theorem 4.6. Let $2 < \alpha < +\infty$. Let T be (α,∞)–bounded with representing operator measure E and $\alpha/\alpha+\epsilon$–atomistic dominating measure. Then T satisfies the pointwise ergodic theorem if and only if $\lim_{n\to+\infty} E_g\{\, 0 < |\xi| < 2^{-n} \,\} = 0$, a.s. for all $g \in L^\alpha(P)$. Let T be (α,β)–bounded, $\beta < +\infty$, T satisfies the pointwise ergodic theorem.

Proof. As in Theorem 3.2, 3.3, and 3.7. ∎

For operators between Hilbert spaces, Theorem 4.6 has the following interesting particular case. Let $T : L^2(P) \longrightarrow L^2(P)$ be invertible with also $\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty$, and let $T^0 = I$. By a result of Sz.–Nagy [Sz.N], there exists a unitary operator U and an invertible Hermitian operator Q such that $T = Q^{-1}UQ$. Hence, for any $g \in L^2(P)$, $\|\sum_{n=-N}^{N} a_n T^n g\|_2^2 \le \|Q^{-1}\|^2\|Q\|^2\|\sum_{n=-N}^{N} a_n U^n g\|_2^2 = K\int_{-\pi}^{\pi} |\sum_{n=-N}^{N} a_n e^{in\theta}|^2 d\|Eg\|^2 \le K\|g\|_2^2 \sup_\theta |\sum_{n=-N}^{N} a_n e^{in\theta}|^2$, and the group T is $(2,\infty)$-bounded. In other words, power bounded and invertible power bounded discrete groups on $L^2(\mathcal{P})$ are exactly the Fourier transforms of the σ-additive, spectral measures from the Borel σ-algebra $\mathcal{B}(]-\pi,\pi])$ to $B(L^2)$ (the $(2,\infty)$-bounded defining property, i.e., $\|\sum_{n=-N}^{N} a_n T^n g\|_2 \leq K\|g\|_2 \sup_\theta |\sum_{n=-N}^{N} a_n e^{in\theta}|$ trivially gives $\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty$). Combining this observation with the previous results we get.

Corollary 4.7. Let T be an invertible bounded linear operator on $L^2(\mathcal{P})$ such that $\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty$. Then, T is $(2,\infty)$-bounded with associated spectral measure E, and it satisfies the pointwise ergodic theorem if and only if $\lim_{n\to+\infty} E_g\{\, 0 < |\xi| < 2^{-n} \,\} = 0$, a.s. for all $g \in L^2(\mathcal{P})$.

Remark 4.8. It is not clear to us how Corollary 4.7 relates to the usual positivity assumption encountered in ergodic theory. In particular, we do not know how it relates to de la Torre's [T] ergodic theorem, i.e., we do not understand why for T positive ($Tf \geq 0$ whenever $f \geq 0$) the condition $\lim_{n\to+\infty} E_g\{\, 0 < |\xi| < 2^{-n} \,\} = 0$, a.s. is always satisfied. A better understanding (a characterization?) of the effects of positivity on the spectral measure is certainly the key to this problem. Unfortunately, Theorem 4.6 does not give any information about, say, the ergodicity of the isometries in $L^\alpha(\mathcal{P})$, $\alpha \neq 2$. It is shown in [CH] that the class of moving averages of Lévy motion (for which the shift exists and is an invertible isometry) and the (α,∞)-bounded class are disjoints (the results in [CH] continue to hold for shifts T such that $\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty$). However, isometries and more generally power bounded and invertible power bounded operators on $L^\alpha(\mathcal{P})$ do admit another type of spectral representation (see Berkson and Gillespie [BG]) which for $\alpha = 2$ corresponds to $(2,\infty)$-boundedness. This spectral representation might help to study the ergodicity of such operators on $L^\alpha(\mathcal{P})$, $\alpha \neq 2$.

It is clear that there are various potential extensions and generalizations of the above results. These include, for example, the local ergodic theorem or the pointwise ergodic theorem for "pseudo" Hermitian operators (see [K]), more generally for operators for which some kind of spectral representation with respect to a non orthogonally scattered operator measure holds. Except for random fields, to which our last section is devoted, we only state as a sample, a result for which the passage from unitary to power bounded and invertible power bounded operators is rather safe.

For unitary operators Jajte [J] proved that the convergence of the ergodic averages and the existence of the ergodic Hilbert transform are equivalent. Combining Jajte's arguments as well as the methods presented here, this equivalence holds more generally:

Corollary 4.9. Let T be an invertible bounded linear operator on $L^2(\mathcal{P})$ such that $\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty$. Then, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n g$ exists a.s. $\mathcal{P}$, for every $g \in L^2(\mathcal{P})$ if and only if $\lim_{N\to\infty} \sum_{0<|n|\leq N} T^n g/n$ exists a.s. $\mathcal{P}$, for every $g \in L^2(\mathcal{P})$.

5. The Spectral SLLN for Random Fields

We assume to the end of this paper that $X = \{X_t\}_{t\in\mathbb{R}^m}$ is a strongly continuous, norm bounded random field. It is easily seen that the univariate (α,β)–boundedness, $1 \leq \alpha \leq 2$, definition given in [H3] carries over to the case of fields, in fact, even to the LCA framework). Essentially as in [H3], it also follows that X is (α,β)–bounded if and only if $X_t = \lim_{\lambda=(\lambda_1,\ldots\lambda_m)\to+\infty} \int_{-\lambda_1}^{\lambda_1} \cdots \int_{-\lambda_m}^{\lambda_m} (1- \frac{|\xi_1|}{\lambda_1})e^{it_1\xi_1} \cdots (1- \frac{|\xi_m|}{\lambda_m})e^{it_m\xi_m}\, dZ(\xi_1,\ldots\xi_m)$ in $L^\alpha(\mathcal{P})$, uniformly on the compacts, where $Z : \mathcal{B}_0(\mathbb{R}^m) \longrightarrow L^0(\mathcal{P})$ has finite (α,β)-variation. The multidimensional version of Grothendieck's inequality continue to hold (see [P]).

Lemma 5.1. Let the random field X be (α,∞)bounded with associated random measure Z. Then, there exists a finite positive measure ν on $\mathbb{R}^m$ such that

$$\left\|\int_{\mathbb{R}^m} f dZ\right\|_\alpha \le \left(\int_{\mathbb{R}^m} |f|^2 d\nu\right)^{1/2}, \tag{5.1}$$

for all $f \in L^2(\nu)$.

For fields, averaging is always more delicate than for processes. Throughout, we follow Gaposhkin [G1], denote by $|A_\rho|$ the volume of A_ρ, and study the averages, $\sigma_\rho X(\omega) = \frac{1}{|A_\rho|}\int_{A_\rho} X(t, \omega)dt$, $\rho > 0$, $\omega \in \Omega$, where $X(t, \omega) = X(t_1, t_2, \cdots, t_m, \omega)$, $dt = dt_1 dt_2 \ldots dt_m$, and where the A_ρ satisfy the following three conditions:

(i) For each ρ, A_ρ is a bounded convex body containing the origin.

(ii) For $0 < \rho_0 \le \rho < \rho'$, $A_\rho \subset A_{\rho'}$ and $\frac{|A_{\rho'}| - |A_\rho|}{|A_\rho|} \le K\frac{\rho' - \rho}{\rho}$.

(iii) There exist two positive constants K_1 and K_2 such that the length $d(\rho)$ of any chord of A_ρ passing through the origin satisfies $K_1\rho \le d(\rho) \le K_2\rho$, $\rho \ge \rho_0 > 0$.

It is clear that n–dimensional spheres of radius ρ with center at the origin satisfy the above three conditions. This is also true of n–dimensional cubes centered at the origin with side of length ρ. Rectangles with do not flatten out also satisfy these conditions. We finally say that X satisfies the SLLN whenever $\lim_{\rho\to+\infty} \sigma_\rho X = 0$, with probability one. For $\xi \in \mathbb{R}^m$ we set $|\xi| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_m^2)^{1/2}$, and then have.

Theorem 5.2. Let the random field X be (α,∞)–bounded, with random measure Z and $\alpha/2$–atomistic dominating measure. Then, X satisfies the SLLN if and only if for almost all ω, $\lim_{p\to+\infty} Z\{|\xi| < 2^{-p}\}(\omega) = 0$.

Proof. The proof requires only adjustments from the univariate results and so will only be sketched. Again, $\sigma_\rho X = (\sigma_\rho X - \sigma_n X) + (\sigma_n X - \sigma_{2^p} X) + (\sigma_{2^p} X - Z\{|\xi| < 2^{-p}\}) + Z\{|\xi| < 2^{-p}\}$. Since, $\|X_t\|_\alpha \le K$, using condition (ii) we easily see that for $n < \rho \le n+1$,

$$\|\sigma_\rho X - \sigma_n X\|_\alpha^\alpha \le K\{|1/|A_\rho| - 1/|A_n||^\alpha \|\sigma_n X\|_\alpha^\alpha + |A_\rho - A_n|^\alpha/|A_n|^\alpha\} \le Kn^{-\alpha}.$$

Hence, with probability one, $\lim_{n\to\infty} \sup_{n<\rho\le n+1} |\sigma_\rho X - \sigma_n X| = 0$. For the third bracket, let $K_\rho(\xi) = \frac{1}{|A_\rho|}\int_{A_\rho} e^{it.\xi}dt$, where $e^{it.\xi} = \sum_{j=1}^{m} e^{it_j\xi_j}$, then clearly $\sigma_{2^p}X = \int_{\mathbb{R}^m} K_{2^p}(\xi)dZ(\xi)$. Since (ii) and (iii) give $|K_{2^p}(\xi) - 1| \le K2^p|\xi|$ whenever $|\xi| < 2^{-p}$, and $|K_{2^p}(\xi)| \le K/(2^p|\xi|)$ for $|\xi| \ge 2^{-p}$, breaking $\mathbb{R}^m$ into $\{|\xi| < 2^{-p}\}$ and $\{|\xi| \ge 2^{-p}\}$, we get using Lemma 5.1, $\sum_{p=1}^{\infty} \mathcal{E}|\sigma_{2^p}X - Z\{|\xi| < 2^{-p}\}|^\alpha < +\infty$. Hence, for almost all ω, $\lim_{p\to+\infty} \sigma_{2^p}X - Z\{|\xi| < 2^{-p}\} = 0$. For the middle bracket, we note that Lemma 3.1 with its notations continue to hold, we apply Lemma 5.1 and we also break $\mathbb{R}^m$ into four pieces: $\{|\xi| \le 2^{-p-1}\}$, $\{2^{-p-1}<|\xi|\le 2^{-p+k}\}$, $\{2^{-p+k}<|\xi| \le 1\}$, $\{|\xi| > 1\}$. For $|\xi| \le 2^{-p-1}$, we have using (ii) and (iii) $|K_{a_k}(\xi) - K_{a_{k-1}}(\xi)| \le K|\xi||a_k - a_{k-1}|$. For, $2^{-p-1}<|\xi|\le 2^{-p+k}$ we have by (i) $|K_{a_k}(\xi) - K_{a_{k-1}}(\xi)| \le K|a_k - a_{k-1}|/|a_{k-1}|$. For, $2^{-p+k}<|\xi| \le 1$, we use the previous inequality and the fact that (i) and (iii) gives $|K_{a_k}(\xi)| \le K|a_k\xi|^{-1}$ to get $|K_{a_k}(\xi) - K_{a_{k-1}}(\xi)| \le K|a_k - a_{k-1}|/|a_{k-1}||\xi a_{k-1}|$. Finally, when $|\xi| > 1$, again (i) and (iii) gives $|K_{a_k}(\xi)| \le K|a_k\xi|^{-1}$, hence $|K_{a_k}(\xi) - K_{a_{k-1}}(\xi)| \le K|a_{k-1}\xi|^{-1}$. Now, as in the proof of Lemma 3.1, these estimate lead, with the atomistic assumption, to four convergent series and the result follows. ∎

Remark 5.3. The requirement on the A_ρ's are just set to ensure that the kernels K_ρ do satisfy the right estimates and so, for any average for which such estimates hold, Theorem 5.2 continue to be true. It is also clear that Theorem 3.3, 3.5, and 3.7 admit multidimensional versions and that as in [G1], the sequence $\{2^{-p}\}$ can be replaced by more general sequences.

BIBLIOGRAPHY

[BG] E. Berkson, T.A. Gillespie, Steckin's theorem, transference and spectral decompositions, *J. Func. Anal. 70 (1987), 140–170.*

[CHW] S. Cambanis, C. Hardin and A. Weron, Ergodic properties of certain stationary stable processes, *Stochastic Process. Appl. 24 (1987), 1–18.*

[CH] S. Cambanis, C. Houdré, Stable noise: moving averages versus Fourier transforms, University of North Carolina Center for Stochastic Processes Technical Report No. 297, May 1990.

[D] D. Dehay, Strong law of large numbers for weakly harmonizable processes, *Stochastic Process. Appl. 24 (1987), 259–267.*

[F] M. Feder, On power bounded operators and the pointwise ergodic property, *Proc. AMS 83 (1981), 349–353.*

[GK] L. Gàl, J. Koksma, Sur l'ordre de grandeur des fonctions sommables, *Proc. Konink. Ned. Akad. v. Wetensh. 53 (1950), 638–653.*

[G1] V.F. Gaposhkin, Criteria for the strong law of large numbers for some classes of second–order stationary processes and homogeneous random fields, *Th. Probability Appl. 22 (1977), 286–310.*

[G2] V.F. Gaposhkin, A theorem on the convergence almost everywhere of measurable functions, and its applications to sequences of stochastic integrals, *Math USSR Sbornik 33 (1977), 1–17.*

[G] R. K. Getoor, The shift operator for non–stationary stochastic processes, *Duke Math. J. 23 (1956), 175–187.*

[H1] H. Helson, The Spectral Theorem, *LN 1227 Springer–Verlag, Berlin, Heidelberg 1986.*

[H1] C. Houdré, Stochastic processes as Fourier integrals and dilation of vector measures. *Bull. Am. Math. Soc. 21 (2) (1989), 281–285.*

[H2] C. Houdré, Harmonizability, V–boundedness, (2,P)–boundedness of stochastic processes, *Probab. Th. Rel. Fields 84 (1990), 39–45.*

[H3] C. Houdré, Linear Fourier and stochastic analysis, University of North Carolina Center for Stochastic Processes Technical Report No. 258, April 1989 (to appear in *Probab. Th. Rel. Fields 1990*).

[J] R. Jajte, On the existence of the ergodic Hilbert transform, *Ann. Probab. 15 (1987), 831–835.*

[K] I. Kluvánek, Characterization of Fourier–Stieljes transformations of vector and operator valued measures. *Czech. Math. J. 17 (1967), 261–277.*

[P] G. Pisier, Factorization of linear operators and geometry of Banach spaces. *CBMS Regional Conf. Ser. in Math., no. 60. Providence, R.I.: Amer. Math. Soc. 1986.*

[R] J. Rousseau-Egelé, La loi forte des grands nombres pour les processus harmonisables, *Ann. Inst. H. Poincaré B 15 (1979), 175–186.*

[Sz.N] B. Sz. Nagy, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math. (Szeged) 11 (1947), 152–157.*

[T] A. de la Torre, Simple proof of the maximal ergodic theorem, *Can. J. Math. 28 (1976), 1073–1075.*

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